OPERATIONS RESEARCH

Linear Programming:

Applications and Model Formulation

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LP Approach

To concentrate on the development and application of specific operations research techniques is to determine the optimal choice among several courses of action, including the evaluation of specific numerical values (if required). For this, it is required to construct (or formulate) a mathematical model. The formulation of a model is important because it represents the essence of a situation or system requiring competent decision-analysis. The term *formulation* referred to the process of converting the verbal description and numerical data into mathematical expressions which represents the relationship among relevant decision variable (or factors), objective and restrictions (constraints) on the use of resources.

. . . LP Approach

Linear Programming (LP) is a mathematical modeling technique useful for the allocation of ‘scarce’ or ‘limited’ resources, such as labour, material, machine, time, warehouse space, capital, energy, etc., to several competing activities, such as products, services, jobs, new equipment, projects, etc., on the basis of a given criterion of optimality. The phrase *scarce resources* means resources that are not available in infinite quantity during the planning period. The criterion of optimality, generally is either performance, return on investment, profit, cost, utility, time, distance, etc.

. . . LP Approach

It is important to understand the meaning of the words, *linear* and *programming.* The word *linear* refers to linear relationship among variables in a model. Thus, a given change in one variable will always cause a resulting proportional change in another variable.

For example, doubling the investment on a certain project will exactly double the rate of return. The word *programming* refers to modeling and solving a problem mathematically that involves the economic allocation of limited resources by choosing a particular *course of action* or *strategy* among various alternative strategies to achieve the desired objective.

General Structure of LP Model

Decision variables (Activities): Evaluation of various alternatives (courses of action) is required for arriving at the optimal value of objective function. Obviously, if there are no alternatives to select from, then there is no need to apply the LP technique. The evaluation of various alternatives is guided by the nature of objective function and availability of resources. For this, we pursue certain activities (also called decision variables), usually denoted by x_1, x_2, . . ., x_n. The value of these activities represents the extent to which each of these is performed. For example, in a product-mix manufacturing, the management may use LP to decide how many units of each of the product to manufacture by using its limited resources such as personnel, machinery, money, material, etc.

. . . LP Model

The value of certain variables may or may not be under the decision-maker's control. If values are under the control of the decision-maker, then such variables are said to be *controllable* otherwise *uncontrollable*. These decision variables, usually interrelated in terms of consumption of limited resources, require simultaneous solutions. In an LP model all decision variables are continuous, controllable and non-negative. That is, $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$.

. . . LP Model

The objective function: The objective (goal) function of each LP problem is expressed in terms of decision variables to optimize the criterion of optimality (also called measure-of-performance) such as profit, cost, revenue, distance, etc. The general objective function of LP model is expressed as:

Optimize (maximize or minimize) $Z = c_1x_1 + c_2x_2 + . . . + c_nx_n$

where Z is the measure-of-performance variable, which is a function of x_1, x_2, . . ., x_n. Quantities c_1, c_2, . . ., c_n are parameters that represent the contribution of a unit of the respective variable x_1, x_2, . . ., x_n to the measure-of-performance Z.

. . . LP Model

The constraints: There are always certain limitations (or constraints) on the use of resources, e.g. labour, machine, raw material, space, money, etc., that limit the degree to which an objective can be achieved. Such constraints must be expressed as linear equalities or inequalities in terms of decision variables. The solution of an LP model must satisfy these constraints.

. . . LP Model

Assumptions of LP Model

- **Certainty:** In all LP models, it is assumed, that all model parameters such as availability of resources, profit (or cost) contribution of a unit of decision variable and consumption of resources by a unit of decision variable must be known and may be constant. In some cases, these may be either random variables represented by a known distribution (general or may statistical) or may tend to change, then the given problem can be solved by a stochastic LP model or parametric programming.

. . . LP Model

- **Additivity:** The value of the objective function and the total amount of each resource used (or supplied), must be equal to the sum of the respective individual contributions (profit or cost) by decision variables. For example, the total profit earned from the sale of two products A and B must be equal to the sum of the profits earned separately from A and B. Similarly, the amount of a resource consumed for producing A and B must be equal to the sum of resources used for A and B individually.
- **Linearity *(or proportionality)*:** The amount of each resource used (or supplied) and its contribution to the profit (or cost) in objective function must be proportional to the value of each decision variable. For example, if production of one unit of a product uses 5 hours of a particular resource, then making 3 units of that product uses 3×5 = 15 hours of that resource.

. . . LP Model

- **Divisibility (or continuity):** The solution values of decision variables are allowed to assume continuous values. For instance, it is possible to produce 6.254 thousand gallons of a solvent in a chemical company, so these variables are divisible. But in contrast, it is not desirable to produce 2.5 machines. Such variables are not divisible and therefore must be assigned integer values. Hence, if any of the variable can assume only integer values or are limited to discrete number of values, LP model is no longer applicable, but an integer programming model may be applied to get the desired values.

Advantages Linear Programming

- Linear programming helps in attaining the optimum use of productive resources. It also indicates how a decision-maker can employ his productive factors effectively by selecting and distributing (allocating) these resources.
- Linear programming technique improves the quality of decisions. The decision-making approach of the user of this technique becomes more objective and less subjective.
- Linear programming technique provides possible and practical solutions since there might be other constraints operating outside the problem which must be taken into account. Just because we can produce so many units does not mean that they can be sold. Thus, necessary modification of its mathematical solution is required for the sake of convenience to the decision-maker.

. . . Linear Programming

- Highlighting of bottlenecks in the production processes is the most significant advantage of this technique. For example, when a bottleneck occurs, some machines cannot meet demand while other remains idle for some of the time.
- Linear programming also helps in re-evaluation of a basic plan for changing conditions. If conditions change when the plan is partly carried out, they can be determined so as to adjust the remainder of the plan for best results.

. . . Linear Programming

Limitations of Linear Programming

- Linear programming treats all relationships among decision variables as linear. However, generally, neither the objective functions nor the constraints in real-life situations concerning business and industrial problems are linearly related to the variables.
- While solving an LP model, there is no guarantee that we will get integer valued solutions. For example, in finding out how many men and machines would be required to perform a particular job, a non-integer valued solution will be meaningless. Rounding off the solution to the nearest integer will not yield an optimal solution. In such cases, integer programming is used to ensure integer value to the decision variables.
- Linear programming model does not take into consideration the effect of time and uncertainty. Thus, the LP model should be defined in such a way that any change due to internal as well as external factors can be incorporated.

. . . Linear Programming

- Sometimes large-scale problems can be solved with linear programming techniques even when assistance of computer is available. For it, the main problem can be fragmented into several small problems and solving each one separately.
- Parameters appearing in the model are assumed to be constant but in real-life situations, they are frequently neither known nor constant.
- It deals with only single objective, whereas in real-life situations we may come across conflicting multi-objective problems. In such cases, instead of the LP model, a goal programming model is used to get satisfactory values of these objectives.

General Mathematical Model of Linear Programming Problem

The general linear programming problem (or model) with n decision variables and m constraints can be stated as:

Find the values of decision variables $x_1, x_2, \ldots, x_n$ so as to

Optimize (Max. or Min.) $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

subject to the linear constraints,

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \;(\le, =, \ge)\; b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \;(\le, =, \ge)\; b_2 \\
&\quad \vdots \qquad\quad \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \;(\le, =, \ge)\; b_m
\end{aligned}
$$

and $x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0.$

where, the c_j's are coefficients representing the per unit contribution of decision variable x_j, to the value of objective function. The a_{ij}'s are called the *technological coefficients or input-output coefficients and* represents the amount of resource, say *i* consumed per unit of variable (activity) x_j. In the given constraints, the a_{ij}'s can be positive, negative or zero. The b_i represents the *total availability of the ith resource.* The term resource is used in a very general sense to include any numerical value associated with the right-hand side of a constraint. It is assumed that for all *i*. However, if any b_i<0, then both sides of constraint *i* can be multiplied by –1 to make b_i > 0 and reverse the inequality of the constraint.

In the general LP problem, the expression ($\leq, =, \geq$) means that in any specific problem each constraint may take only one of the three possible forms:

(i) less than or equal to ($\leq$)
(ii) equal to (=)
(iii) greater than or equal to ($\geq$)

Guidelines on Linear Programming Model Formulation

Step 1: Identify the decision variables

- Express each constraint in words. For this first see whether the constraint is of the form (at least as large as), or of the form (no larger than) or = (exactly equal to).
- Then express the objective function verbally.
- Steps (a) and (b) should then allow you to verbally identify the decision variables.

If there are several decision alternatives available, then to identify the decision variables you have to ask yourself the question: *What decisions must be made in order to optimize the objective function?*

Having accomplished Step 1(a) through (c) decide the symbolic notation for the decision variables and specify units of measurement. Such specification of units of measurement would help in interpreting the final solution of the LP problem.

Step 2: Identify the problem data

For solving a problem, we need to identify the problem data so as to provide the actual values for the decision variables. For this, we need to enumerate all types of information with respect to the given problem to determine these values of the decision variables. These quantities constitute the problem data. It may be noted that the decision-maker can control values of the variables, but cannot control the values of the data.

Step 3: Formulate the constraints

- Express the constraints verbally in terms of requirements and availability of each resource.
- Convert the verbal expression of the constraints imposed by the resource availability as linear equality or inequality in terms of the decision variables defined in Step 1.

These constraints are the conditions that the decision variable must satisfy to constitute an acceptable (feasible) solution. These constraints typically arise due to physical limitations, management-imposed restrictions, external restrictions, logical restrictions on individual variables, implied relationships among variables, etc. Wrong formulation can lead to either solutions which are not feasible or excluding some solutions which are really feasible and possibly optimal.

Step 4: Formulate the objective function

Identify whether the objective function is to be maximized or minimized. Then express it verbally, such as, maximize total profit/cost and then convert it into a linear mathematical expression in terms of decision variables multiplied by their profit or cost contributions.

Examples of LP Model Formulation

Example 1: A manufacturing company is engaged in producing three types of products: A, B and C. The production department produces, each day, components sufficient to make 50 units of A, 25 units of B and 30 units of C. The management is confronted with the problem of optimizing the daily production of products in assembly department where only 100 man-hours are available daily to assemble the products. The following additional information is available.

Type of Product	Profit Contribution per Unit of Product (Rs)	AssemblyTime per Product (hrs)
A	12	0.8
B	20	1.7
C	45	2.5

The company has a daily order commitment for 20 units of products A and a total of 15 units of products B and C. Formulate this problem as an LP model so as to maximize the total profit.

LP model formulation: The data of the problem is summarized as follows:

Resources/Constraints	*Product Type*			*Total*
	A	*B*	*C*	
Production capacity (units)	50	25	30	
Man-hours per unit	0.8	1.7	2.5	100
Order commitment (units)	20	15 (both for B and C)		
Profit contribution (Rs/unit)	12	20	45	

Decision variables: Let x_1, x_2 and x_3 = number of units of products A, B and C to be produced, respectively.

The LP model

Maximize (total profit) $Z = 12x_1 + 20x_2 + 45x_3$

subject to the constraints

(i) Labour and materials constraints

$$\begin{aligned}
0.8x_1 + 1.7x_2 + 2.5x_3 &\leq 100 \\
x_1 &\leq 50 \\
x_2 &\leq 25 \\
x_3 &\leq 30
\end{aligned}$$

(ii) Order commitment constraints

$$\begin{aligned}
x_1 &\geq 20 \\
x_2 + x_3 &\geq 15
\end{aligned}$$

and $x_1, x_2, x_3 \geq 0$

Example 2: A company has two grades of inspectors 1 and 2, who are to be assigned for a quality control inspection. It is required that at least 2,000 pieces be inspected per 8-hour day. Grade 1 inspector can check pieces at the rate of 40 per hour, with an accuracy of 97 per cent. Grade 2 inspector checks at the rate of 30 pieces per hour with an accuracy of 95 per cent.

The wage rate of a Grade 1 inspector is Rs 5 per hour while that of a Grade 2 inspector is Rs 4 per hour. An error made by an inspector costs Rs 3 to the company. There are only nine Grade 1 inspectors and eleven Grade 2 inspectors available in the company. The company wishes to assign work to the available inspectors so as to minimize the total cost of the inspection. Formulate this problem as an LP model so as to minimize daily inspection cost.

LP model formulation: The data of the problem is summarized as follows:

	Inspector	
	Grade 1	*Grade 2*
Number of inspectors	9	11
Rate of checking	40 pieces/hr	30 pieces/hr
Inaccuracy in checking	1 – 0.97 = 0.03	1 – 0.95 = 0.05
Cost of inaccuracy in checking	Rs 3/piece	Rs 3/piece
Wage rate/hour	Rs 5	Rs 4

Duration of inspection = 8 hrs per day

Total pieces which must be inspected = 2,000

Decision variables: Let x_1 and x_2 = number of Grade 1 and 2 inspectors to be assigned for inspection, respectively.

The LP model

Hourly cost of each of Grade 1 and 2 inspectors can be computed as follows:

Inspector Grade 1: Rs $(5 + 3 \times 40 \times 0.03)$ = Rs 8.60
Inspector Grade 2: Rs $(4 + 3 \times 30 \times 0.05)$ = Rs 8.50

Minimize (daily inspection cost) $Z = 8\,(8.60x_1 + 8.50x_2) = 68.80x_1 + 68.00x_2$

subject to the constraints

(i) Total number of pieces that must be inspected in an 8-hour day constraint

$$8 \times 40x_1 + 8 \times 30x_2 \geq 2000$$

(ii) Number of inspectors of Grade 1 and 2 available constraint

$$x_1 \leq 9; \quad x_2 \leq 11$$

and $$x_1,\ x_2 \geq 0$$

Example 3: A pharmaceutical company produces two pharmaceutical products: A and B. Production of both products requires the same process, I and II. The production of B results also in a by-product C at no extra cost. The product A can be sold at a profit of Rs 3 per unit and B at a profit of Rs 8 per unit. Some of this by-product can be sold at a unit profit of Rs 2, the remainder has to be destroyed and the destruction cost is Re 1 per unit. Forecasts show that only up to 5 units of C can be sold. The company gets 3 units of C for each unit of B produced. The manufacturing times are 3 hours per unit for A on process I and II, respectively, and 4 hours and 5 hours per unit for B on process I and II, respectively. Because the product C results from producing B, no time is used in producing C. The available times are 18 and 21 hours of process I and II, respectively. Formulate this problem as an LP model to determine the quantity of A and B which should be produced, keeping C in mind, to make the highest total profit to the company.

LP model formulation:

The data of the problem is summarized as follows:

Constraints/Resources	*Time (hrs) Required by*			*Availability*
	A	*B*	*C*	
Process I	3	4	–	18 hrs
Process II	3	5	–	21 hrs
By-product ratio from B	–	1	3	5 units (max. units that can be sold)
Profit per unit (Rs)	3	8	2	

Decision variables: Let

x_1, x_2 = units of product A and B to be produced, respectively

x_3, x_4 = units of product C to be produced and destroyed, respectively.

The LP model

Maximize (total profit) $Z = 3x_1 + 8x_2 + 2x_3 - x_4$

subject to the constraints

(I) Manufacturing constraints for product A and B

$$3x_1 + 4x_2 \leq 18$$

$$3x_1 + 5x_2 \leq 21$$

(ii) Manufacturing constraints for by-product C

$$x_3 \leq 5$$

$$-3x_2 + x_3 + x_4 = 0$$

and $x_1, x_2, x_3, x_4 \geq 0$

Example 4: A company, engaged in producing tinned food, has 300 trained employees on the rolls, each of whom can produce one can of food in a week. Due to the developing taste of the public for this kind of food, the company plans to add to the existing labour force by employing 150 people, in a phased manner, over the next five weeks. The newcomers would have to undergo a two-week training programme before being put to work. The training is to be given by employees from among the existing ones and it is known that one employee can train three trainees. Assume that there would be no production from the trainers and the trainees during training period as the training is off-the-job. However, the trainees would be remunerated at the rate of Rs 300 per week, the same rate as for the trainers.

The company has booked the following orders to supply during the next five weeks:

Week	:	1	2	3	4	5
No. of cans	:	280	298	305	360	400

Assume that the production in any week would not be more than the number of cans ordered for so that every delivery of the food would be ‘fresh’.

Formulate this problem as an LP model to develop a training schedule that minimizes the labour cost over the five-week period.

LP model formulation: The data of the problem is summarized as:

(i) Cans supplied

Week :	1	2	3	4	5
Number :	280	298	305	360	400

(ii) Each trainee has to undergo a two-week training.

(iii) One employee is required to train three trainees.

(iv) Every trained worker producing one can/week but no production from trainers and trainees during training.

(v) Number of employees to be employed = 150

(vi) The production in any week not to exceed the cans required.

(vii) Number of weeks for which newcomers would be employed: 5, 4, 3, 2, 1.

From the given information you may observe following facts:

- Workers employed at the beginning of the first week would get salary for all the five weeks; those employed at the beginning of the second week would get salary for four weeks and so on.
- The value of the objective function would be obtained by multiplying it by 300 because each person would get a salary of Rs 300 per week.
- Inequalities have been used in the constraints because some workers might remain idle in some week(s).

Decision variables:

Let x_1, x_2, x_3, x_4 and x_5 = number of trainees appointed in the beginning of week 1, 2, 3, 4 and 5, respectively.

The LP model

Minimize (total labour force) $Z = 5x_1 + 4x_2 + 3x_3 + 2x_4 + x_5$

subject to the constraints

(i) Capacity constraints

$$300 - \frac{x_1}{3} \geq 280 \qquad 300 - \frac{x_1}{3} - \frac{x_2}{3} \geq 298$$

$$300 + x_1 - \frac{x_2}{3} - \frac{x_3}{3} \geq 305 \qquad 300 + x_1 + x_2 - \frac{x_3}{3} - \frac{x_4}{3} \geq 360$$

$$300 + x_1 + x_2 + x_3 - \frac{x_4}{3} - \frac{x_5}{3} \geq 400$$

(ii) New recruitment constraint

$$x_1 + x_2 + x_3 + x_4 + x_5 = 150$$

and $\quad x_1, x_2, x_3, x_4, x_5 \geq 0$

Example 5: A plastic manufacturer has 1,200 boxes of transparent wrap in stock at one factory and another 1,200 boxes at its second factory. The manufacturer has orders for this product from three different retailers, in quantities of 1,000, 700 and 500 boxes, respectively. The unit shipping costs (in rupees per box) from the factories to the retailers are as follows:

	Retailer I	*Retailer II*	*Retailer III*
Factory A	14	13	11
Factory B	13	13	12

Determine a minimum cost shipping schedule for satisfying all demands from current inventory. Formulate this problem as an LP model.

LP model formulation: Given that the total number of boxes available at factory A and B = total number of boxes required by retailers 1, 2 and 3.

Decision variables: x_1, x_2 and x_3 = number of boxes to be sent from factory A to retailer 1; factory B to retailer 2 and factory C to retailer 3, respectively.

	Number of Boxes to be Sent		
	Retailer 1	*Retailer 2*	*Retailer 3*
Factory A	x_1	x_2	$1{,}200 - (x_1 + x_2)$
Factory B	$1{,}000 - x_1$	$700 - x_2$	$1{,}000 - [(1{,}000 - x_1) + (700 - x_2)]$

The LP model

Minimize (total distance)

$$Z = 14x_1 + 13x_2 + 11\,(1{,}200 - x_1 - x_2) + 13\,(1{,}000 - x_1) + 13\,(700 - x_2) + 12\,(x_1 - x_2 - 700) = 2x_1 + x_2 + 26{,}900$$

subject to the constraints

$$x_1 + x_2 \le 1{,}200$$

$$x_1 \le 1{,}000;\ x_2 \le 700;$$

$$x_1 + x_2 \ge 7{,}000.$$

and $x_1, x_2 \ge 0$

Example 6: A company produces two types of sauces: A and B. These sauces are both made by blending two ingredients X and Y. A certain level of flexibility is permitted in the formulae of these products. Indeed, the restrictions are that (i) B must contain no more than 75 per cent of X, and (ii) A must contain no less than 25 per cent of X, and no less than 50 per cent of Y. Up to 400 kg of X and 300 kg of Y could be purchased. The company can sell as much of these sauces as it produces at a price of Rs 18 for A and Rs 17 for B. The X and Y cost Rs 1.60 and 2.05 per kg, respectively. The company wishes to maximize its net revenue from the sale of these sauces. Formulate this problem as an LP model.

LP model formulation x_1, x_2 = kg of sauces A and B to be produced, respectively

y_1, y_2 = kg of ingredient X used to make sauces A and B, respectively

y_3, y_4 = kg of ingredient Y used to make sauces A and B, respectively

The LP model

Maximize $Z = 18x_1 + 17x_2 - 1.60\,(y_1 + y_2) - 2.05\,(y_3 + y_4)$

subject to the constraints

$$
\begin{aligned}
y_1 + y_3 - x_1 &= 0 \\
y_2 + y_4 - x_2 &= 0 \\
\left.\begin{array}{l} y_1 + y_2 \leq 400 \\ y_3 + y_4 \leq 300 \end{array}\right\} &\text{ (Purchase)} \\
\left.\begin{array}{l} y_1 - 0.25x_1 \geq 0 \\ y_2 - 0.50x_2 \geq 0 \end{array}\right\} &\text{ (Sauce A)} \\
y_2 - 0.75x_2 \geq 0 &\text{ (Sauce B)}
\end{aligned}
$$

and $x_1, x_2, y_1, y_2, y_3, y_4 \geq 0.$

Example 7: A businessman is opening a new restaurant and has budgeted Rs 8,00,000 for advertisement in the coming month. He is considering four types of advertising:

- 30 second television commercials
- 30 second radio commercials
- Half-page advertisement in a newspaper
- Full-page advertisement in a weekly magazine which will appear four times during the coming month.

The owner wishes to reach families with income both over and under Rs 50,000. The amount of exposure to families of each type and the cost of each of the media is shown below:

Media	Cost of Advertisement (Rs)	Exposure to Families with Annual Income Over Rs 50,000	Exposure to Families with Annual Income Under Rs 50,000
Television	40,000	2,00,000	3,00,000
Radio	20,000	5,00,000	7,00,000
Newspaper	15,000	3,00,000	1,50,000
Magazine	15,000	1,00,000	1,00.000

To have a balanced campaign, the owner has determined the following restrictions:

- no more than four television advertisements
- no more than four advertisements in the magazine
- no more than 60 per cent of all the advertisements in newspaper and magazine
- there must be at least 45,00,000 exposures to families with incomes over Rs 50,000.

Formulate this problem as an LP model to determine the number of each type of advertisement to pursue so as to maximize the total number of exposures.

LP model formulation

Let x_1, x_2, x_3 and x_4 = number of television, radio, newspaper, magazine advertisements to pursue, respectively.

The LP model

Maximize (total number of exposures of both groups) Z

$= (2,00,000 + 3,00,000)\ x_1 + (5,00,000 + 7,00,000)\ x_2 + (3,00,000 + 1,50,000)x_3 + (1,00,000 + 1,00,000)\ x_4$

$= 5,00,000\ x_1 + 12,00,000\ x_2 + 4,50,000\ x_3 + 2,00,000\ x_4$

subject to the constraints

- Available budget constraint

  $40,000x_1 + 20,000x_2 + l5,000x_4 + 5,000x_4 \leq 8,00,000$

- Maximum television advertisement constraint

  $x_1 \leq 4$

- Maximum magazine advertisement constraint

  $x_4 \leq 4$ (because magazine will appear only four times in the next month)

- Maximum newspaper and magazine advertisement constraint

$$\frac{x_3 + x_4}{x_1 + x_2 + x_3 + x_4} \leq 0.60$$

or $\quad - 0.6x_1 - 0.6x_2 + 0.4x_3 + 0.4x_4 \leq 0$

- Exposure to families with income over Rs 50,000 constraint

$2{,}00{,}000x_1 + 5{,}00{,}000x_2 + 3{,}00{,}000x_3 + 1{,}00{,}000x_4 \geq 45{,}00{,}000$

and $\quad x_1, x_2, x_3, x_4 \geq 0.$

Example 7: An advertising agency is preparing an advertising campaign for a group of agencies. These agencies have decided that their target customers should have the following characteristics with importance (weightage) as given below.

	Characteristics	*Weightage* (%)
Age	25–40 years	20
Annual income	Above Rs 60,000	30
Female	Married	50

The agency has made a careful analysis of three media and has compiled the following data.

Data Item	*Media*		
	Women's Magazine (%)	*Radio* (%)	*Television* (%)
Reader characteristics			
(i) Age: 25–40 years	80	70	60
(ii) Annual income: Above Rs 60,000	60	50	45
(iii) Females/Married	40	35	25
Cost per advertisement (Rs)	9,500	25,000	1,00,000
Minimum number of advertisement allowed	10	5	5
Maximum number of advertisement allowed	20	10	10
Audience size (1000s)	750	1,000	1,500

The budget for launching the advertising campaign is of Rs 5,00,000. Formulate this problem as an LP model for the agency to maximize the total expected effective exposure.

LP model formulation:

Let x_1, x_2 and x_3 = number of advertisements made using advertising media: women's magazines, radio and television, respectively.

The effectiveness coefficient corresponding to each of the advertising media is calculated as follows:

Media	*Effectiveness Coefficient*
Women's magazine	0.80 (0.20) + 0.60 (0.30) + 0.40 (0.50) = 0.54
Radio	0.70 (0.20) + 0.50 (0.30) + 0.35 (0.50) = 0.46
Television	0.60 (0.20) + 0.45 (0.30) + 0.25 (0.50) = 0.38

The coefficient of the objective function, i.e. effective exposure for all the three media employed can be computed as follows:

$$\textit{Effective exposure} = \textit{Effectiveness coefficient} \times \textit{Audience size}$$

where effectiveness coefficient is a weighted average of audience characteristics.

Effective exposure of each media is as follows:

Women's magazine	= 0.54 × 7,50,000 = 4,05,000
Radio	= 0.46 × 10,00,000 = 4,60,000
Television	= 0.38 × 15,00,000 = 5,70,000

The LP model

Maximize (effective exposure) $Z = 4{,}05{,}000x_1 + 4{,}60{,}000x_2 + 5{,}70{,}000x_3$

subject to the constraints

(i) Budget constraint

$$9{,}500x_1 + 25{,}000x_2 + 1{,}00{,}000x_3 \leq 5{,}00{,}000$$

(ii) Minimum number of advertisements allowed constraints

$$x_1 \geq 10; \quad x_2 \geq 5; \quad x_3 \geq 5$$

(iii) Maximum number of advertisements allowed constraints

$$x_1 \leq 20; \quad x_2 \leq 10; \quad x_3 \leq 10$$

and $x_1, x_2, x_3 \geq 0.$

Example 8: XYZ is an investment company. To aid in its investment decision, the company has developed the investment alternatives for a 10-year period, as given in the following table. The return on investment is expressed as an annual rate of return on the invested capital. The risk coefficient and growth potential are subjective estimates made by the portfolio manager of the company. The terms of investment is the average length of time period required to realize the return on investment as indicated.

Investment Alternative	*Length of Investment*	*Annual Rate of Return (Year)*	*RiskCoefficient Return (%)*	*GrowthPotential*
A	4	3	1	0
B	7	12	5	18
C	8	9	4	10
D	6	20	8	32
E	10	15	6	20
F	3	6	3	7
Cash	0	0	0	0

The objective of the company is to maximize the return on its investments.

The guidelines for selecting the portfolio are:

- The average length of the investment for the portfolio should not exceed 7 years.
- The average risk for the portfolio should not exceed 5.
- The average growth potential for the portfolio should be at least 10%.
- At least 10% of all available funds must be retained in the form of cash at all times.

Formulate this problem as an LP model to maximize total return.

LP model formulation: Let x_j = proportion of funds to be invested in the jth investment alternative (j = 1, 2, . . ., 7)

The LP model

Maximize (total return) $Z = 0.03x_1 + 0.12x_2 + 0.09x_3 + 0.20x_4 + 0.15x_5 + 0.06x_6 + 0.00x_7$

subject to the constraints

(i) Length of investment constraint

$$4x_1 + 7x_2 + 8x_3 + 6x_4 + 10x_5 + 3x_6 + 0x_7 \leq 7$$

(ii) Risk level constraint

$$x_1 + 5x_2 + 4x_3 + 8x_4 + 6x_5 + 3x_6 + 0x_7 \leq 5$$

(iii) Growth potential constraint

$$0x_1 + 0.18x_2 + 0.10x_3 + 0.32x_4 + 0.20x_5 + 0.07x_6 + 0x_7 \geq 0.10$$

(iv) Cash requirement constraint

$$x_7 \geq 0.10$$

(v) Proportion of funds constraint

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 1$$

and $\quad x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$

Example 8: An investor has three investment opportunities available at the beginning of each for the next 5 years, and also has a total of Rs 5,00,000 available for investment at the beginning of the first year. Financial characteristics of the three investment alternatives is presented below:

Investment Alternative	Allowable Size of Initial Investment (Rs.)	Return (%)	Timing of Return	Immediate Reinvestment Possible?
1	1,00,000	19	1 year later	yes
2	unlimited	16	2 years later	yes
3	50,000	20	3 years later	yes

The investor wishes to determine the investment plan that will maximize the amount of money which can be accumulated by the beginning of the 6th year in the future. Formulate this problem as an LP model to maximize total return.

LP model formulation:

x_{ij} = amount to be invested in investment alternative, i (i = 1, 2, 3) at the beginning of the year j (j =1, 2, . . ., 5)

y_j = amount not invested in any of the investment alternatives in period j

The LP model

Minimize (total return) $Z = 1.19x_{15} + 1.16x_{24} + 1.20x_{33} + y_5$

subject to the constraints

(i) Yearly cash flow constraints

$$x_{11} + x_{21} + x_{31} + y_1 = 5{,}00{,}000 \text{ (year 1)}$$

$$-y_1 - 1.19x_{11} + x_{12} + x_{22} + x_{32} + y_2 = 0 \text{ (year 2)}$$

$$-y_2 - 1.16x_{21} - 1.19x_{12} + x_{23} + x_{23} + x_{33} + y_3 = 0$$

$$-y_3 - 1.20x_{31} - 1.16x_{22} - 1.19x_{13} + x_{14} + x_{24} + x_{34} + y_4 = 0 \text{ (year 4)}$$

$$-y_4 - 1.20x_{32} - 1.16x_{23} - 1.19x_{14} + x_{15} + x_{25} + x_{35} + y_5 = 0 \text{ (year 5)}$$

(ii) Size of investment constraints

$$x_{11} \le 1{,}00{,}000 \qquad x_{31} \le 50{,}000$$
$$x_{12} \le 1{,}00{,}000 \qquad x_{32} \le 50{,}000$$
$$x_{13} \le 1{,}00{,}000 \qquad x_{33} \le 50{,}000$$
$$x_{14} \le 1{,}00{,}000 \qquad x_{34} \le 50{,}000$$
$$x_{15} \le 1{,}00{,}000 \qquad x_{35} \le 50{,}000$$

and $x_{ij}, y_j \ge 0$ for all *i* and *j*.

Example 7: ABC manufacturing company wishes to develop a monthly production schedule for the next three months. Depending upon the sales commitments, the company can either keep the production constant, allowing fluctuation in inventory; or inventories can be maintained at a constant level, with fluctuating production. Fluctuating production makes overtime work necessary, the cost of which is estimated to be double the normal production cost of Rs 12 per unit. Fluctuating inventories result in an inventory carrying cost of Rs 2 per unit/month. If the company fails to fulfill its sales commitment, it incurs a shortage cost of Rs 4 per unit/month. The production capacities for the next three months are shown below:

Month	Production Capacity (units)		Sales
	Regular	Overtime	(units)
1	50	30	60
2	50	0	120
3	60	50	140

Formulate this problem as an LP model to minimize the total production cost.

LP model formulation:

Month	Production Capacity		Sales
	Regular	Overtime	
1	50	30	60
2	50	0	120
3	60	50	40

Normal production cost	:	Rs 12 per unit
Overtime cost	:	Rs 24 per unit
Inventory carrying cost	:	Rs 2 per unit per month
Shortage cost	:	Rs 4 per unit per month

Assume five sources of supply: three regular and two overtime (because second month overtime production is zero) production capacities. The demand for the three months will be the sales during these months.

All supplies against order have to be made and can be made in the subsequent month if not possible during the month of order, with additional cost equivalent to shortage cost, i.e. in month 2. The cumulative production of months 1 and 2 in regular and overtime is 130 units while the orders are for 180 units. This balance can be supplied during month 3 at an additional production cost of Rs 4.

	M_1	M_2	m_3	Production (supply)
M_1	12	14	16	50
M_2	16	12	14	50
M_3	20	16	12	60
m_1(OT)	24	26	28	30
M_2(OT)	32	28	24	50
Sales (demand)	60	120	40	

Decision variables: Let x_{ij} = amount of commodity sent from source of supply i (i = 1, 2, . . ., 5) to destination j (j = 1, 2, 3)

The LP model

Minimize (total cost) $Z = 12x_{11} + l4x_{12} + 16x_{13} + 16x_{21} + l2x_{22} + 14x_{23} + 20x_{31} + 16x_{32} + 12x_{33} + 24x_{41} + 26x_{42} + 28x_{43} + 32x_{51} + 28x_{52} + 24x_{53}$

subject to the constraints

(i) Production (supply) constraints

$$x_{11} + x_{12} + x_{13} = 50$$
$$x_{21} + x_{22} + x_{23} = 50$$
$$x_{31} + x_{32} + x_{33} = 60$$
$$x_{41} + x_{42} + x_{43} = 30$$
$$x_{51} + x_{52} + x_{53} = 30$$

(ii) Sales (demand) constraints

$$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} = 160$$
$$x_{12} + x_{22} + x_{32} + x_{42} + x_{52} = 120$$
$$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} = 140$$

and $x_{ij} \geq 0$ for all i and j

Example 8: Evening shift resident doctors in a government hospital work five consecutive days and have two consecutive days off. Their five days of work can start on any day of the week and the schedule rotates indefinitely. The hospital requires the following minimum number of doctors working:

Sun	Mon	Tues	Wed	Thus	Fri	Sat
35	55	60	50	60	50	45

No more than 40 doctors can start their five working days on the same day. Formulate this problem as an LP model to minimize the number of doctors employed by the hospital.

LP model formulation: Let x_j = number of doctors who start their duty on day j (j = 1, 2, . . ., 7) of the week.

The LP model

Minimize (total number of doctors) $Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$

subject to the constraints

$$
\begin{aligned}
x_1 + x_4 + x_5 + x_6 + x_7 &\geq 35 \\
x_2 + x_5 + x_6 + x_7 + x_1 &\geq 55 \\
x_3 + x_6 + x_7 + x_1 + x_2 &\geq 60 \\
x_4 + x_7 + x_1 + x_2 + x_3 &\geq 50 \\
x_5 + x_1 + x_2 + x_3 + x_4 &\geq 60 \\
x_6 + x_2 + x_3 + x_4 + x_5 &\geq 50 \\
x_7 + x_3 + x_4 + x_5 + x_6 &\geq 45 \\
x_j &\leq 40
\end{aligned}
$$

and $x_j \geq 0$ for all j.

Example 9: A machine tool company conducts on the job training programme for machinists. Trained machinists are used as teachers in the programme in the ratio of one for every ten trainees. The training programme lasts for one month. From past experience it has been found that out of ten trainees hired, only seven complete the programme successfully and the rest are released.

Trained machinists are also needed for machining and company's requirement for the next three months is as follows: January 100, February 150 and March 200. In addition, the company requires 250 machinists by April. There are 130 trained machinists available at the beginning of the year. Pays per month are:

Each trainee	:	Rs 4,400
Each trained machinist (machining and teaching)	:	Rs 4,900
Each trained machinist idle	:	Rs 4,700

Formulate this problem as an LP model to minimize the cost of hiring and training schedule and the company's requirements.

LP model formulation

x_1, x_2 = trained machinist teaching and idle in January, respectively

x_3, x_4 = trained machinist teaching and idle in February, respectively

x_5, x_6 = trained machinist teaching and idle in March, respectively

The LP model

Minimize (total cost) Z = Cost of training program (teachers and trainees) + Cost of idle machinists + Cost of machinists doing machine work (constant)

$$= 4{,}400\,(10x_1 + 10x_3 + 10x_5) + 4{,}900\,(x_1 + x_3 + x_5) + 4{,}700\,(x_2 + x_4 + x_6)$$

subject to the constraints

(i) Total trained machinists available at the beginning of January

= Number of machinists doing machining + Teaching + Idle

$$130 = 100 + x_1 + x_2$$

$$x_1 + x_2 = 30$$

(ii) Total trained machinists available at the beginning of February

= Number of machinists in January + Joining after training program

$$130 + 7x_1 = 150 + x_3 + x_4$$

$$7x_1 - x_3 - x_4 = 20$$

In January there are $10x_1$ trainees in the program and out of those only $7x_1$ will become trained machinists.

(iii) Total trained machinists available at the beginning of March

= Number of machinists in January + Joining after training programme in January and February

$$130 + 7x_1 + 7x_3 = 200 + x_5 + x_6$$

$$7x_1 + 7x_3 - x_5 - x_6 = 70$$

(iv) Company requires 250 trained machinists by April

$$130 + 7x_1 + 7x_3 + 7x_5 = 250$$

$$7x_1 + 7x_3 + 7x_5 = 120$$

and $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Example 10: The super bazzar in a city daily needs between 22 and 30 workers depending on the time of day. The rush hours are between noon and 2 PM. The table below indicates the number of workers needed at various hours when the bazar is open.

Time Period	*Number of Workers Needed*
9 AM – 11 AM	22
11 AM – 1 PM	30
1 PM – 3 PM	25
3 PM – 5 PM	23

The super bazar now employs 24 full-time workers, but needs a few part-time workers also. A part- time worker must put in exactly 4 hours per day, but can start any time between 9 AM and 1 PM. Full- time workers work from 9 AM to 5 PM but are allowed an hour for lunch (half of the full-timers eat at 12 noon, the other half at 1 PM). Full-timers thus provide 35 hours per week of productive labour time.

The management of the super bazar limits part-time hours to a maximum of 50 per cent of the day’s total requirement.

Part-timers earn Rs 28 per day on the average, while full-timers earn Rs. 90 per day in salary and benefits on the average. The management wants to set a schedule that would minimize total manpower costs.

Formulate this problem as an LP model to minimize total daily manpower cost.

LP model formulation:

y = full-time workers

x_j = part-time workers starting at 9 AM, 11 AM and 1 PM, respectively (j = 1, 2, 3)

The LP model

Minimize (total daily manpower cost) $Z = 90y + 28(x_1 + x_2 + x_3)$

subject to the constraints

$y + x_1 \geq 22$ [9 AM–11 AM need]

$\frac{1}{2}y + x_1 + x_2 \geq 30$ [11 AM – 1 PM need]

$\frac{1}{2}y + x_2 + x_3 \geq 25$ [1 PM – 3 PM need]

$y + x_3 \geq 23$ [3 PM – 5 PM need]

$y \leq 24$ [Full-timers available]

$$4\,(x_1 + x_2 + x_3) \leq 0.50\,(22 + 30 + 25 + 23)$$

and $y, x_j \geq 0$ for all j.